ADVANCED NUMBER THEORY
MID-TERM EXAM

This exam is of **30 marks** and is **2 hours long**. Please **read all the questions carefully** and **do not cheat**. Please feel free to use whatever theorems you have learned in class after stating them clearly. Please sign the following statement:

I have not used any unfair or illegal means to answer any of the questions in this exam. I have answered this paper honestly and truthfully.

Name: **Signature:**

1. Prove that the Fourier expansion of the Eisenstein series around the cusp ∞ is 10

$$G_k(z) = 2\zeta(2k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

where $q = \exp(2\pi i n z)$ and B_k are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum B_k \frac{x^k}{k!}$$

You may use the product formula $\sin(\pi x) = \pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)$.

2. Let $G = \mathbb{Z}/N\mathbb{Z}$ and $\xi = e^{\frac{2\pi i}{N}}$. Let $f : G \to \mathbb{C}$ be a function. Define

$$\hat{f}(a) = \sum_{b\in B} f(b)\xi^{-ab}$$

Show that 3

$$f(b) = \frac{1}{N}\sum_{a\in G}\hat{f}(a)\xi^{ab}$$

3. Let χ be a primitive Dirichlet character mod N and

$$g(\chi) = \sum_{a\in G}\chi(a)\xi^a$$

Show that 10

$$L(s,\chi) = \frac{1}{N}g(\chi)\sum_{a\in G}\bar{\chi}(a)\sum_{n=1}^{\infty}\frac{\xi^{-an}}{n^s}$$

4. Use this to obtain a formula for $L(1,\chi)$ in terms of log. 2

5. Show that the image of any homomorphism $\phi : SL_2(\mathbb{Z}) \to \mathbb{C}^*$ lies in the subgroup of the 12^{th} roots of unity. 5